Mathematics 101 Problem Set 1 Solutions

1(a) Show from First principles that any constant function $f(x) = C$ satisfies $f'(x) = 0$ (meaning that $f'(x)$ is the *zero function*).

(b) Find necessary and sufficient conditions on a, b, c, and d to ensure that

$$f(x) = \frac{ax + b}{cx + d}$$

is a constant function.

(c) By polynomial division show that

$$f(x) = \frac{a}{c} + \frac{bc - ad}{c(cx + d)}$$

and hence draw the same conclusion as in (b). What is

$$\lim_{x\to\infty} \frac{ax + b}{cx + d}?$$

(d) Find the equation of the inverse function $f^{-1}(x)$, where $f(x)$ is as in 1(b).

Solution (a)

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{C - C}{h} = \lim_{h\to 0}(0) = 0.$$

(b) $f(x)$ will be constant if and only if $f'(x) = 0$, which is to say, using the Quotient rule of differentiation:

$$\frac{a(cx + d) - c(ax + b)}{(cx + d)^2} \equiv 0 \Leftrightarrow \frac{ad - bc}{(cx + d)^2} \equiv 0 \Leftrightarrow ad = bc.$$

Comment: the symbol $\equiv$ here means 'is identically equal to'. This emphasises that $f(x) = 0$ for all values of x, distinguishing it from an equality in an equation such as $2x + 3 = 8$, which is true only for some values of x.

(c) The first line in the division gives the quotient of $\frac{c}{a}$ and then by subtraction we obtain:

$$f(x) = \frac{c}{a} + \frac{bc - ad}{c(cx + d)}.$$

Written in this form we see that $f(x)$ is constant if and only if $ad - bc = 0$, in which case $f(x) \equiv \frac{c}{a}$ for all x. We should check what happens if $a = 0$. In that case $f(x) = \frac{b}{cx+d}$ and then $ad = 0$ implies $bc = 0$, whence either $b = 0$ (in which case $f(x) \equiv 0$) or $c = 0$, in which case $f(x) \equiv \frac{b}{d}$. Also

$$\lim_{x\to\infty} \frac{ax+b}{cx+d} = \lim_{x\to\infty} \frac{a+\frac{b}{x}}{c+\frac{d}{x}} = \frac{a+0}{c+0} = \frac{a}{c}.$$

(d) To find the rule for the inverse function, we solve $y = f(x)$ for x, which in this case gives:

$$y(cx+d) = ax+b \Leftrightarrow x(a-cy) = dy - b \Rightarrow x = \frac{dy-b}{a-cy}.$$

$$\therefore f^{-1}(x) = \frac{dx-b}{a-cx}.$$

2(a) Find a function $f(x)$ such that:

$$f(2x+3) = x^2+1.$$

[Hint: we have an equation of the form $f(g(x)) = h(x)$ and we want $f(x)$, so replace x by $g^{-1}(x)$ as then we get $f(x) = h(g^{-1}(x))$.]

(b) Find a linear function $f(x) = ax + b$ such that $f(f(x)) = 2x + 1$.

Solution (a) Taking the hint, we note that here $g(x)$ is the function $y = 2x+3$. Inverting gives $x = \frac{y-3}{2}$ so that $g^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$. We then replace x by $g^{-1}(x)$ in $f(2x+3)$ and this gives:

$$f(x) = (\frac{1}{2}x - \frac{3}{2})^2 + 1 = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{9}{4} + 1$$

$$\therefore f(x) = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}.$$

Check:

$$f(2x+3) = \frac{1}{4}(2x+3)^2 - \frac{3}{2}(2x+3) + \frac{13}{4}$$

$$= x^2 + 3x + \frac{9}{4} - 3x - \frac{9}{2} + \frac{13}{4} = x^2 + 1.$$

(b)

$$f(f(x)) = a(ax+b) + b = 2x + 1$$

$$\Rightarrow a^2x + b(1+a) = 1.$$

Equating coefficients gives $a^2 = 2$ so that $a = \pm\sqrt{2}$. Then

$$b = \frac{1}{1+a} = \frac{1}{1\pm\sqrt{2}} = \frac{1\mp\sqrt{2}}{(1\pm\sqrt{2})(1\mp\sqrt{2})} = \frac{1\mp\sqrt{2}}{1-2} = \pm\sqrt{2} - 1.$$

$$\therefore f(x) = \sqrt{2}x + \sqrt{2} - 1 \text{ or } f(x) = -\sqrt{2}x - (\sqrt{2}+1).$$

Check: (2nd solution)

$$f(f(x)) = -\sqrt{2}(-\sqrt{2}x - \sqrt{2} - 1) - \sqrt{2} - 1 = 2x + 2 + \sqrt{2} - \sqrt{2} - 1 = 2x + 1.$$

3. Find from First Principles the derivatives of the functions with the following rules:
(a) $2x+1$; (b) $1-x^2$; (c) $\frac{1}{1+x}$.

Solutions (a) Let $f(x) = 2x+1$. Then

$$f'(a) = \lim_{h\to 0} \frac{2(a+h)+1-(2a+1)}{h} = \lim_{h\to 0} \frac{2a+2h+1-2a-1}{h} = \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2.$$

$$\therefore f'(x) \equiv 2.$$

(b)

$$f'(a) = \lim_{h\to 0} \frac{(1-(a+h)^2)-(1-a^2)}{h} = \lim_{h\to 0} \frac{1-a^2-2ah-h^2-1+a^2}{h} = \lim_{h\to 0} \frac{-2ah-h^2}{h}$$

$$= -\lim_{h\to 0}(2a+h) = -2a.$$

$$\therefore f'(x) = -2x.$$

(c)

$$f'(a) = \lim_{h\to 0} \frac{\frac{1}{1+a+h} - \frac{1}{1+a}}{h} = \lim_{h\to 0} \frac{1+a-1-a-h}{h(1+a)(1+a+h)} = -\lim_{h\to 0} \frac{1}{(1+a)(1+a+h)} = -\frac{1}{(1+a)^2}.$$

$$\therefore f'(x) = -\frac{1}{(1+x)^2}.$$

4. Find the equations of the two tangents to the curve $y = x^2$ that pass through the point $(2,0)$.

Each tangent line L has an equation of the form $y - y_0 = m(x - x_0)$, where (x_0, y_0) is an point on the line. Since L passes through $(2,0)$ we thus have $y - 0 = y = m(x-2)$. Also L, since it is a tangent, touches the curve at some point $P(a, a^2)$. The slope m of L matches the gradient of the curve at P, so that $m = 2a$, and also, since the tangent passes through P, $a^2 = m(a-2)$. This gives $a^2 = 2a(a-2)$, so either $a = 0$ or

$$a = 2(a-2) \Rightarrow a = 2a - 4 \Rightarrow a = 4.$$

Therefore L is given by either $y = 0$ (the x-axis) or

$$y = 8(x-2) \Leftrightarrow y = 8x - 16.$$

5. Find the derivative of the cosine function from First Principles.

Solution

$$f'(a) = \lim_{h\to 0} \frac{\cos(a+h) - \cos a}{h} = \lim_{h\to 0} \frac{\cos a \cos h - \sin a \sin h - \cos a}{h} = \lim_{h\to 0} \frac{\cos a(\cos h - 1) - \sin a \sin h}{h}$$

$$= \cos a \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin a \lim_{h\to 0} \frac{\sin h}{h} = (\cos a)(0) - \sin a(1) = -\sin a.$$

Alternatively,

$$f'(a) = \lim_{h\to 0} \frac{\cos(a+h) - \cos a}{h} = \lim_{h\to 0} \frac{-2\sin(a+\frac{h}{2})\sin\frac{h}{2}}{h}$$

$$= -\lim_{h\to 0} \frac{\sin\frac{h}{2}}{h/2} \cdot \lim_{h\to 0} \sin(a + \frac{h}{2}) = -1 \cdot \sin a = -\sin a.$$

$$\therefore f'(x) = -\sin x.$$

6. Prove that $f(x) = cx + d$ $(c, d \in \mathbb{R},\ c \neq 0)$ is continuous by taking $\delta = \frac{\epsilon}{|c|}$ in the definition of continuity.

Solution Suppose $|x - a| < \delta = \frac{\epsilon}{|c|}$. Then

$$|f(x+a) - f(x)| = |c(x+a) + d - (cx+d)| = |cx + ca + d - cx - d|$$

$$= |c(a-x)| = |c| \cdot |x-a| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon,$$

as required to show that $f(x) = cx + d$ is continuous everywhere.

7. Solve the following inequalities:

(a) $\frac{x-1}{1-3x} > 7$; (b) $|3x - 4| \leq 20$; (c) $|-2(1-5x)| > 8$.

Solutions
(a)

$$\frac{x-1}{1-3x} > 7 \Rightarrow (x-1)(1-3x) > 7(1-3x)^2 \Leftrightarrow (1-3x)\big((x-1) - 7(1-3x)\big) > 0$$

$$(1-3x)\big(x - 1 - 7 + 21x\big) > 0 \Leftrightarrow (1-3x)(22x-8) > 0$$

$$\Leftrightarrow (3x-1)(11x-4) < 0.$$

The quadratic on the left has roots $x = \frac{1}{3}$ and $x = \frac{4}{11} > \frac{1}{3}$. Hence the inequality is satisfied strictly between these roots, that is

$$\frac{1}{3} < x < \frac{4}{11}.$$

(b) We re-write $|3x-4| \le 20$ as

$$-20 \le 3x - 4 \le 20$$
$$\Leftrightarrow -16 \le 3x \le 24$$
$$-\frac{16}{3} \le x \le 8.$$

(c)

$$|-2(1-5x)| = |2(5x-1)| = |2| \cdot |5x-1| = 2|5x-1| > 8 \Leftrightarrow |5x-1| > 4$$
$$\Leftrightarrow 5x - 1 < -4 \text{ or } 5x - 1 > 4$$
$$\Leftrightarrow 5x < -3 \text{ or } 5x > 5$$
$$\Leftrightarrow x < -\frac{3}{5} \text{ or } x > 1.$$

8. Given that $|x-2| \le 3$ and $|y+2| < 1$, what is the set of all possible values of $2x+3y$?

Solution

$$|x-2| \le 3 \Leftrightarrow -3 \le x - 2 \le 3 \Leftrightarrow -1 \le x \le 5 \Leftrightarrow -2 \le 2x \le 10;$$
$$|y+2| < 1 \Leftrightarrow -1 < y + 2 < 1 \Leftrightarrow -3 < y < -1 \Leftrightarrow -9 < 3y < -3.$$
$$\therefore -11 < 2x + 3y < 7.$$

9. Find all values of x such that

$$|x| + |-5x| = 10.$$

Solution

$$|x| + |-5x| = |x| + |-5| \cdot |x| = |x| + 5|x| = 6|x| = 10$$
$$\Rightarrow |x| = \frac{10}{6} = \frac{5}{3}$$
$$\therefore x = \pm\frac{5}{3}.$$

10. Solve

$$|2x+1| = 1 + |1-3x|.$$

Solution

$$2x + 1 = \pm(1 + |3x-1|)$$
$$\Leftrightarrow (x \ge \frac{1}{3} \,\&\, 2x{+}1 = \pm(1{+}3x{-}1) = \pm 3x \text{ or } x \le \frac{1}{3} \,\&\, 2x + 1 = \pm(1 - 3x + 1) = \pm(2 - 3x)|$$
$$\Leftrightarrow (x \ge \frac{1}{3} \,\&\, (x = 1 \text{ or } x = -\frac{1}{5})) \text{ or } (x \le \frac{1}{3} \,\&\, (x = \frac{1}{5} \text{ or } x = 3)$$
$$\Leftrightarrow x = 1 \text{ or } x = \frac{1}{5}.$$