MA181 Discrete Mathematics

Lecture 2 – Set Laws

Let A, B and C denote arbitrary sets. Sets obey the following laws, all of which, with the possible exceptions of Laws 4 and 10, are (fairly) obvious.

Laws of Algebra of Sets

1. Idempotent Laws:

1a $A \cup A = A$ 1b $A \cap A = A$.

2. Associative Laws:

2a $(A \cup B) \cup C = A \cup (B \cup C)$ 2a $(A \cap B) \cap C = A \cap (B \cap C)$.

3. Commutative Laws:

3a $A \cup B = B \cup A$ 3b $A \cap B = B \cap A$

4. Distributive Laws:

4a $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 4b $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

5 & 6. Identity Laws:

5a $A \cup \emptyset = A$ 5b $A \cap \mathcal{U} = A$ 6a $A \cup \mathcal{U} = \mathcal{U}$ 6b $A \cap \emptyset = \emptyset$.

7. Involution Law: $(A^c)^c = A$.

8 & 9. Complement Laws:

8a $A \cup A^c = \mathcal{U}$ 8b $A \cap A^c = \emptyset$ 9a $\mathcal{U}^c = \emptyset$ 9b $\emptyset^c = \mathcal{U}$.

10. De Morgan's Laws:

10a $(A \cup B)^c = A^c \cap B^c$ 10b $(A \cap B)^c = A^c \cup B^c$.

In order to convince oneself of a statement of equality of two sets (i.e. that they are the same thing) it is often a good idea to represent the expressions involved by means of Venn Diagrams. However, any real proof that two sets X and Y are equal must be verified by showing that every element x in X is also in Y (this shows that $X \subseteq Y$) and likewise every element of Y also lies in X (thus showing that $Y \subseteq X$ and so $X = Y$).

Example 2.1

With this in mind let us verify one of the less obvious laws, Law 4a:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{1}$$

First, let $x \in A \cup (B \cap C)$, so $x \in A$ or $x \in B \cap C$. This gives us two cases to test:

(i) If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ so that $x \in (A \cup B) \cap (A \cup C)$.

(ii) If $x \in B \cap C$ then $x \in B$ and so $x \in A \cup B$; similarly $x \in A \cup C$.

Hence $x \in (A \cup B) \cap (A \cup C)$ and so the LHS (left-hand side) of (1) is a subset of the RHS.

Conversely, let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$ which is the same as saying that $x \in A$ *or* B and $x \in A$ *or* C. Once more we split this up into two cases:

(i) If $x \in A$ then certainly $x \in A \cup (B \cap C)$.

(ii) If $x \notin A$ then it must be the case that $x \in B$ and $x \in C$; that is to say $x \in B \cap C$.

Hence in either case, $x \in A \cup (B \cap C)$, and so the RHS of (1) is a subset of the LHS.

These two paragraphs together prove that equation (1) is always true for any sets A, B, and C. Since this is the case we say that (1) is an *identity*.

There is a definite symmetry about the 10 Set Laws that is well worth appreciating. Let E be an equation of set algebra. By the *dual equation* E^* we mean the equation obtained form E by replacing each occurrence of the symbols $\cup, \cap, \mathcal{U}, \emptyset$ by $\cap, \cup, \emptyset, \mathcal{U}$ respectively. Note that $(E^*)^* = E$: the dual statement of a dual statement is the original statement. Hence to say that E^* is the dual of E is equivalent to saying that E is the dual of E^*: in language you will later become familiar with we say that duality is a symmetric relation. Law 7 is merely a statement about complements but all the other Laws occur in dual pairs. The upshot of this is that theorems about sets usually come in pairs.

Principle of Duality Let E be a statement of inclusion or equality of sets involving only the operations of union, intersection and complementation. Then the statement E^* obtained by interchanging unions with intersections, $\mathcal{U}$ with $\emptyset$, and reversing all inclusions is true if E is true and is false if E is false.

The reason why the Principle holds is that, given any proof of a statement E we can turn it into a proof of E^* by use of the dual Law at each stage: for example, if at one line in the proof for E we use Law 4a then at the corresponding stage in the proof for E^* we would use Law 4b.

The operations of union, intersection and complementation of sets are often referred to as *boolean operations* (named after George Boole). The question of deciding whether or not two boolean expressions represent the same set is in general extremely difficult, although drawing Venn diagrams can help if the expressions do not involve too many sets. We can use the Set Laws however to avoid laboriously testing whether an arbitrary member of the LHS is also a member of the RHS and vica-versa, allowing us to give a very clear formal demonstration of the truth of an identity.

Example 2.1 Use the Set Laws to prove that:

$$(A \cap B) \cup (B^c \cap A) = A \tag{2}$$

and state the dual proposition.

Proof

$$\begin{aligned}(A \cap B) \cup (B^c \cap A) &= (A \cap B) \cup (A \cap B^c) \quad \text{law (3b)}\\ &= A \cap (B \cup B^c) \qquad \text{law(4b)}\\ &= A \cap \mathcal{U} \qquad \text{law (8a)}\\ &= A \qquad \text{law (5b).}\end{aligned}$$

Dual: $(A \cup B) \cap (B^c \cup A) = A.$