# MA181 Discrete Mathematics

## Lecture 1 – Sets

These lecture notes contain what is most important in each lecture and should be read *prior* to the lecture. They do however represent only the bare bones. During the lecture itself the content will be augmented with additional examples and explanations about which you may wish to make some supplementary notes of your own.

This course introduces the mathematical ideas with which anyone studying a modern computing or mathematics degree needs to be familiar. The context in which much of modern maths finds itself is that of Sets, and the theory of Sets carries with it its own language which takes some getting used to. For this reason there is a Glossary of Set Theory Symbols that will arise in the course on the back of the first problem sheet. The underlying ideas the symbols represent are however pretty simple.

A *set* is any collection of objects referred to as *elements* or *members* of the set. Sets are generally denoted by capital letters: $A, B, S, X$ etc. while elements are usually denoted by lower case letters: $a, b, c, \cdots, x, y$ etc. The statement '$a$ is an element of $A$' is denoted by $a \in A$; the negation of this is written $a \notin A$. Two sets $A$ and $B$ are said to be *equal* if, and only if, they have the same members, in which case we write $A = B$.

There are two ways of specifying a set. We may *explicitly* define a set by writing down all of its members; for example

$$A = \{-1, 1\}$$

On the other hand we may *implicitly* define a set by specifying that it consists of all objects which satisfy a certain property or properties:

$$B = \{x : x \text{ is an integer and } x^2 = 1\}.$$

Note that in this case it has turned out that $A = B$.

You should note the use of curly brackets when presenting a set. The set $\{1, 2\}$ is not the same as the ordered pair $(1, 2)$ (an idea that we shall explain more carfully later in the lecture).

The order in which the elements happen to be listed is of no account when speaking of a set: the sets $\{Peter, Paul\}$ and $\{Paul, Peter\}$ are the same because they have the same members.

Many of the examples we shall use to illustrate ideas will involve sets of numbers because they are simple to produce but in the real computational world the sets that people are liable to be concerned with have objects of many varied abstract kinds. They are still sets nonetheless and so questions about membership of these sets must be asked and answered using the standard terminology which we are introducing here.

Implicit definition of sets is extremely important but pre-supposes that we already have some context from which we may draw members. In any application of set theory the members of all sets of interest are considered to come from some set $\mathcal{U}$, the *universal set*. This universe of discourse can itself vary – it depends entirely on the kind of things we may wish to consider. At the other extreme the *empty set* or *null set* is the set with no elements, denoted by $\emptyset$, or by $\{\}$. The empty set should not be confused with the number 0 (which is the number of elements in $\emptyset$).

A set $A$ is a *subset* of a set $B$ if each element of $A$ is also an element of $B$; we denote this by $A \subseteq B$ and write $A \not\subseteq B$ to denote the corresponding negation. If $A \subseteq B$ and $A \neq B$ we say that $A$ is a *proper subset* of $B$.

**Set Operations**

The *union* of two sets $A$ and $B$, denoted by $A \cup B$ is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

We use the word 'or' to mean and/or, sometimes called *inclusive or*, and this is the standard usage in mathematics.

The *intersection* of $A$ and $B$, denoted $A \cap B$, is the set of all elements common to both sets:

$$A \cap B = \{x : x \in A \quad \text{and} \quad x \in B\}.$$

**Example 1.1** $A = \{-1, 0, 1, 2\}$, $B = \{1, 2, 4\}$. Then

$$A \cup B = \{-1, 0, 1, 2, 4\}, \quad A \cap B = \{1, 2\}.$$

Note that sets never contain repeats: an element is either in a set or not – it cannot be a member several times over!

The number of elements in a set $A$ is called the *cardinality* of $A$ and is denoted by $|A|$. For instance, for the set $A$ above we have $|A| = 4$.

The *complement* of a set $A$, denoted by $A^c$ or by $\bar{A}$, is the set of all elements of $\mathcal{U}$ which are *not* in $A$. For instance the complement of the set of vowels is the set of consonants (where the universal set here is, of course, the alphabet). The *difference* of sets $B$ and $A$, in that order, denoted by $B \backslash A$ or by $B - A$, is the set of all elements that are in $B$ but not in $A$:

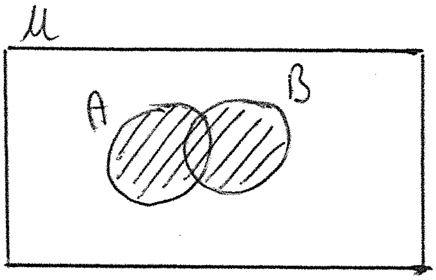
$$B \backslash A = \{x : x \in B,\ x \notin A\}.$$

This is sometimes also called the *relative complement* of $A$ in $B$ - this is because if we take $B = \mathcal{U}$ we get $\mathcal{U} - A$: the complement of $A$ in $\mathcal{U}$ which is the complement of $A$ as introduced in the preceding paragraph. With the sets $A$ and $B$ as in the previous example we have:

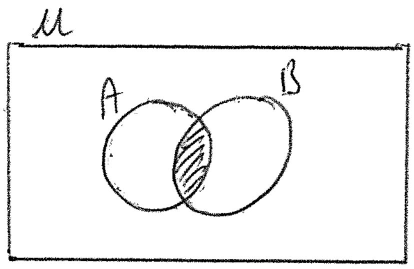
$$B \backslash A = \{4\}; \quad A \backslash B = \{-1, 0\}.$$

The use of the notation $A - B$ for difference is fine, but always bear in mind its true meaning: it has nothing to do with ordinary subtraction. Never be tempted to write $A + B$ when discussing sets as the plus sign has no meaning in this context.

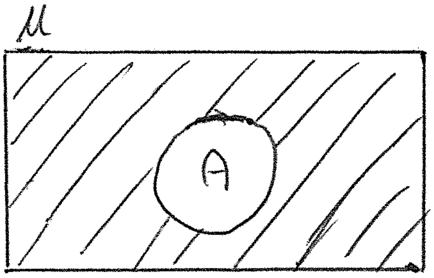
Sets are often pictured as overlapping regions within one universal box and these drawings are called Venn Diagrams. Those illustrating the ideas already introduced are pictured below:



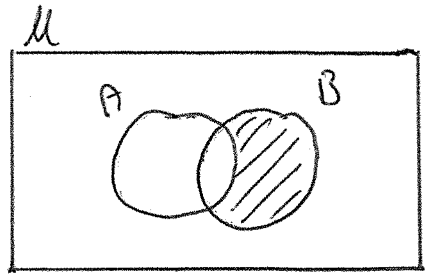

$A \cup B$




$A \cap B$




$\bar{A}$ or $A^c$



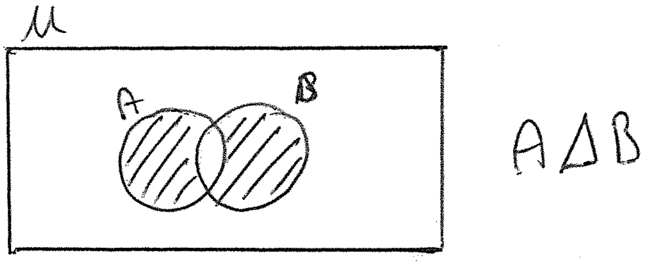

$B \backslash A$

The idea of *exclusive or* (you may have a biscuit or a cake *but not both!*) can be expressed in the context of sets. The *symmetric difference* of two sets $A$ and $B$, denoted $A \bigtriangleup B$, consists of all elements contained in one of the sets $A$ and $B$ but not both:

$$A \bigtriangleup B = \{x : x \in A \text{ or } x \in B \text{ but not both } \}.$$

We now have the notation to express this more succinctly, it fact we can do this in two equivalent ways:

$$A \bigtriangleup B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$




For the sets of Example 1.1 you should check that $A \bigtriangleup B = \{-1, 0, 4\}$.

Two sets $A$ and $B$ are *disjoint* if they have no elements in common: that is to say $A \cap B = \emptyset$.

If $A$ and $B$ are sets then an *ordered pair* from $A$ and $B$ (in that order) is a pair of elements $(a, b)$ where $a \in A$ and $b \in B$. The collection of all ordered pairs from $A$ and $B$ is known as the *direct product* or *cartesian product* (after Rene Descartes) of $A$ and $B$. This set is denoted by $A \times B$. For example if $A = \{-1, 1, 3\}$ and $B = \{-1, 1\}$ then:

$$A \times B = \{(-1, -1), (-1, 1), (1, -1), (1, 1), (3, -1), (3, 1)\}.$$

The idea of an ordered pair you will surely have met before – in school mathematics or indeed in geography – the latitude and longtitude of a point on the globe is an ordered pair. The direct product is just the collection of all ordered pairs under consideration. The $\times$ notation does not of course mean multiply. There is a connection with multiplication however through the cardinality of the set $A \times B$ which does equal $|A| \times |B|$: in the above example $|A| = 3, |B| = 2$ and the direct product $A \times B$ does indeed consist of $2 \times 3 = 6$ ordered pairs.

Note that, as opposed to sets, an ordered pair may have a repeat (for example the pair $(1, 1)$ in the above example) and that, since order is an integral part of an ordered pair, the ordered pairs $(-1, 1)$ and $(1, -1)$ are different ordered pairs, although the sets $\{-1, 1\}$ and $\{1, -1\}$ are equal.

**Our first glossary of set theory notation (more to come later)**

$\in$: element symbol, $a \in A$ means '$a$ is an element of $A$'.

$\subseteq$: subset symbol, $A \subseteq B$ is read '$A$ is a subset of $B$'.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$ means 'the set of all $x$ such that $x$ is an element of $A$ or $x$ is an element of $B$'.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

$A^c = A' = \bar{A}$ is the *complement* of $A$, so $\bar{A} = \{x : x \notin A\}$.

$A \backslash B = A - B = A \cap \bar{B}$.

$\mathcal{U}$: the *universal set.*

$\emptyset$: the *empty* or *null* set.

$A \triangle B$: the *symmetric difference* of $A$ and $B$, which is equal to $(A \cap \bar{B}) \cup (\bar{A} \cap B)$.

$|A|$: the *cardinality* of the set $A$, which the number of elements in $A$.

$A \times B$: the *Cartesian product* or *direct product* of $A$ and $B$, which is the set $\{(a, b) : a \in A, b \in B\}$.