Mathematics 101 Lecture 2

1 Continuous and differentiable functions

As we said in the previous lecture, a function $f(x)$ is continous at $x = a$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a))| < \varepsilon. \tag{1}$$

Writing $x = a + h$ (h may be positive or negative) we may also express continuity of $f(x)$ at a as:

$$\lim_{h \to 0} \big(f(a + h) - f(a)\big) = 0. \tag{2}$$

The expression (2) suggests a connection between differentiability and continuity. Indeed differentiability is 'continuity + smoothness' and we shall show in this lecture that differentiability of $f(x)$ at a certainly implies that $f(x)$ is continuous at a.

Example 2.1.1 Prove that the function $f(x) = x^2$ is continuous at every point $a \in \mathbb{R}$.

Let $\varepsilon > 0$ be given. We need to find a δ which will satisfy (1). We may always assume that $\delta < 1$ for if we find a δ that works for some ε, then the definition will be satisfied when we use any smaller value of δ. In general, a suitable value of δ will depend both on ε and on the particular function with which you are dealing. In practice, you may not be able to tell what a suitable value of δ might be when starting the problem, so let us explore by taking $\delta > 0$ to be arbitrary for the moment and see how small the difference $|f(x) - f(a)|$ will be. Bear in mind that a denotes an arbitrary but fixed value of x. It is acceptable for our choice of δ to be dependent not only on ε but on a as well (but not on the variable x). Now, by the difference of two squares we obtain:

$$|f(x) - f(a)| = |x^2 - a^2| = |(x - a)(x + a)| = |x - a| \cdot |x + a| \tag{3}$$

We want to express $|x + a|$ in terms of $|x - a|$, and so we try writing $x + a = x - a + 2a$ and see where that leads. By the Triangle Inequality we get:

$$|x + a| = |x - a + 2a| \leq |x - a| + |2a| \leq \delta + 2|a| \leq 1 + 2|a|.$$

Hence from (3) we obtain:

$$|f(x) - f(a)| \leq \delta(1 + 2|a|).$$

Since a is fixed we now only have to choose δ so that $\delta < \frac{\varepsilon}{1+2|a|}$ and we have satisfied (1). This proves that $f(x) = x^2$ is continuous at an arbitrary value $x = a$, which is to say that $f(x) = x^2$ is continuous everywhere. □

Example 2.1.2 Prove continuity of the cosine function.

Solution We use the identity $\cos A - \cos B = -2\sin\frac{A-B}{2}\sin\frac{A+B}{2}$. For any chosen $\delta > 0$, we have $-\delta < x - a < \delta$, or in other words:

$$|x - a| < \delta \Leftrightarrow a - \delta < x < a + \delta \Leftrightarrow x = a + h \text{ for some } h : -\delta < h < \delta.$$

Hence the expression $|f(x) - f(a)|$ in the definition of continuity may take the form $|f(a+h) - f(a)|$. In this example we obtain:

$$|\cos(a+h) - \cos a| = |-2\sin\frac{h}{2}\sin(a+\frac{h}{2})| = 2|\sin\frac{h}{2}|\cdot|\sin(a+\frac{h}{2})| \leq 2|\sin\frac{h}{2}| \leq 2\frac{|h|}{2} = |h| < \delta$$

as $0 \leq |\sin x| \leq |x|$ for all x. In particular, for a given $\epsilon > 0$ we may put $\delta = \varepsilon$ and we arrive at

$$|\cos(a+h) - \cos a| < \delta = \varepsilon,$$

thereby showing that $f(x) = \cos x$ is a continuous function.

To find the derivative of the sine function (our next example) we need the fact, proved below, that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. If we take this for granted we may solve the problem by making use of the identity:

$$\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}. \tag{4}$$

Example 2.1.3 Find the derivative of the sine function.

Solution Working from first principles we find the limit for the derivative of $\sin x$ evaluated at $x = a$. For presentational convenience we label our increment from a as $a + 2h$ rather than $a + h$. (You'll see why.) We make use of (4) in the expression for the numerator, as follows.

$$\lim_{h\to 0}\frac{\sin(a+2h) - \sin a}{2h} = \lim_{h\to 0}\frac{2\sin\frac{a+2h-a}{2}\cos\frac{2a+2h}{2}}{2h} = \lim_{h\to 0}\frac{\sin h}{h}\cos(a+h).$$

We now use the fact that the limit of a product is the product of the limits, that the cosine function is continuous, and that $\lim_{h\to 0}\frac{\sin h}{h} = 1$ to conclude that

$$\frac{d(\sin x)}{dx}|_{x=a} = \lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\cos(a+h) = 1\cdot\cos a = \cos a.$$

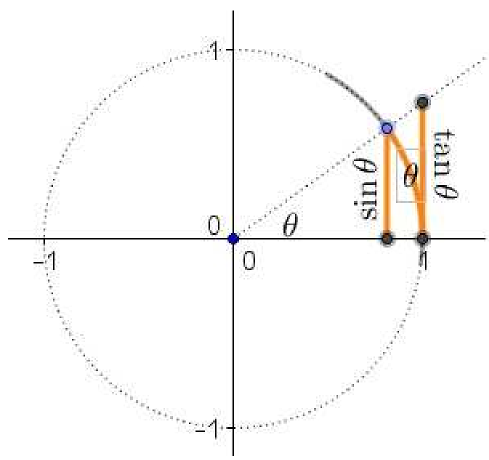
In other words, the derivative of $\sin x$ is $\cos x$.

Theorem 2.1.4

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Proof Consider a unit circle and a small angle θ. Since the vertical distance $\sin\theta$ is less than the corresponding arc θ we have $\sin\theta < \theta$. Since the area of the unit circle is π, the area of the sector bounded by θ is $\frac{\theta}{2\pi}\cdot\pi = \frac{\theta}{2}$. On the other hand the area of the enclosing triangle with vertical side $\tan\theta$ as shown is $\frac{\tan\theta}{2}$. All this leads to $\sin\theta < \theta < \tan\theta$. Dividing by $\sin\theta$ now gives $1 \leq \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$, and then taking reciprocals gives:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1. \tag{5}$$



This shows that (5) holds for $\theta > 0$. However, since $\cos(-\theta) = \cos\theta$ and $\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$ we see that we may replace θ by $-\theta$ in (5) and nothing changes, which is to say that (5) holds for all small values of θ. Finally, letting $\theta \to 0$ we have $\cos\theta \to 1$. Since $\frac{\sin\theta}{\theta}$ is then squeezed between 1 on the right and something approaching 1 on the left, it follows that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. $\square$

Example 2.1.5 A function $f(x)$ being continuous is no guarantee that $f(x)$ is differentiable. For example, consider the function $f(x) = |x|$. It is easy to see that $f(x)$ is both differentiable and continuous at any point $x \neq 0$. Indeed $f'(x) = \pm 1$, with the + sign applying if $x > 0$ and the minus sign applies for negative x. Also $|x|$ is continous as $x = 0$. We need only put $\delta = \varepsilon$ for then $|x - 0| < \delta$ says immediately that $|x| < \varepsilon$, which gives the required conclusion that $|f(x) - f(0)| = ||x| - |0|| < \varepsilon$. Therefore the absolute value function is continuous everywhere. However, the limit in the definition of derivative takes on differing values at $a = 0$ depending on whether x approaches 0 from above (we denote this by $x \downarrow 0$) or x approaches 0 from below, (written as $x \uparrow 0$).

$$\lim_{x\downarrow 0}\frac{|x| - |0|}{x} = \lim_{x\downarrow 0}\frac{x}{x} = 1, \text{ but } \lim_{x\uparrow 0}\frac{|x| - |0|}{x} = \lim_{x\downarrow 0}\frac{-x}{x} = -1.$$

However, differentiablity does always imply continuity. This is simple to show although we need to use one of the rules (which will all be listed in a future lecture) that the limit of a product of two functions as $x \to a$ is the product of the limits of the functions as $x \to a$.

Theorme 2.1.6 If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$. Therefore any differentiable function is continuous.

Proof We are given the existence of $f'(a)$, which is to say that

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a). \tag{6}$$

By putting $x = a + h$, we may re-formulate (6) as

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a) \tag{7}$$

Now to say that $\lim_{x\to a} f(x) = f(a)$ is clearly equivalent to saying that $\lim_{x\to a}(f(x) - f(a)) = 0$, which is what we now verify.

$$\begin{aligned} \lim_{x\to a}(f(x) - f(a)) = \lim_{x\to a}(x-a)\frac{f(x) - f(a)}{x - a} &= \lim_{x\to a}(x-a) \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ &= 0 \cdot f'(a) = 0, \end{aligned}$$

thereby proving that $f(x)$ is continuous at $x = a$. $\square$

Our proof that $(\sin x)' = \cos x$ is a little different from the one in most text books. The alternative method expands the expression $\sin(x+h)$ as $\sin x \cos h + \cos x \sin h$ and to complete the proof you then need to evaluate the following limit, which is our final example.

Example 2.1.7

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0. \tag{8}$$

Solution Since $1 = \cos 0$ we may substitute accordingly and apply the identity for the difference $\cos A - \cos B$ to obtain:

$$\frac{\cos h - \cos 0}{h} = \frac{-2\sin(\frac{h-0}{2})\sin(\frac{h+0}{2})}{h} = -\frac{\sin^2 \frac{h}{2}}{h/2}.$$

$$\therefore \lim_{h\to 0} \frac{\cos h - 1}{h} = -\lim_{h\to 0} \frac{\sin \frac{h}{2}}{h/2} \cdot \lim_{h\to 0} \sin \frac{h}{2} = -1 \times 0 = 0.$$